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COMMON FIXED POINTS OF A PAIR OF SUZUKI $\mathcal{Z}$-CONTRACTION TYPE MAPS IN b-METRIC SPACES

G. V. R. Babu and D. Ratna Babu*

Department of Mathematics,
Andhra University,
Visakhapatnam - 530003, Andhra Pradesh, INDIA

E-mail : gvr_babu@hotmail.com

*Department of Mathematics,
PSCMRCET, Vijayawada - 520001, Andhra Pradesh, INDIA

E-mail : ratnababud@gmail.com

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Abstract: In this paper, we introduce Suzuki $\mathcal{Z}$-contraction type (I) maps, Suzuki $\mathcal{Z}$-contraction type (II) maps, for a pair of selfmaps in b-metric spaces and prove the existence and uniqueness of common fixed points. We draw some corollaries to our results and provide examples in support of our results.

Keywords and Phrases: Common fixed point, b-metric space, b-continuous, Suzuki $\mathcal{Z}$-contraction type maps.

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1. Introduction

Nonlinear analysis plays an important role in many branches of Applied Sciences, for latest works, we refer [13, 20, 24, 25, 26]. Particularly, fixed point theory is a part of nonlinear analysis and its development depends on the generalization of contraction conditions or/and generalization of ambient spaces of the operator under consideration. In 1975, Dass and Gupta [12] established fixed point results using contraction condition involving rational expressions and proved the existence of fixed points in complete metric spaces. In 2008, Suzuki [28] proved two fixed point theorems, one of which is a new type of generalization of the Banach contraction principle and does characterize the metric completeness.

The main idea of b-metric was initiated from the works of Bourbaki [9] and Bakhtin [6]. The concept of b-metric space or metric type space was introduced by Czerwik [10] as a generalization of metric space. Afterwards, many authors studied fixed point theorems for single-valued and multi-valued mappings in b-metric spaces, for more information we refer [2, 7, 8, 11, 15, 18, 19, 27].

In this paper, we denote $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{N}$ is the set of all natural numbers.

Definition 1.1. [10] *Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric if the following conditions are satisfied: for any $x, y, z \in X$;*

(i) $0 \leq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) there exists $s \geq 1$ such that $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space with coefficient s.

Every metric space is a b-metric space with $s = 1$. In general, every b-metric space is not a metric space(Example 4.3, [4]).

Definition 1.2. [8] *Let (X, d) be a b-metric space.*

(i) *A sequence $\{x_n\}$ in X is called b-convergent if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim\limits_{n\to\infty} x_n = x$ and x is unique.*

(ii) *A sequence $\{x_n\}$ in X is called b-Cauchy if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.*

(iii) *A b-metric space (X, d) is said to be a complete b-metric space if every b-Cauchy sequence in X is b-convergent in X.*

In general, a b-metric is not necessarily continuous.

Example 1.3. [14] Let $X = \mathbb{N} \cup \{\infty\}$. We define a mapping $d : X \times X \to [0, \infty)$ as follows:

$$d(m,n) = \begin{cases} 0 & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}| & \text{if one of } m, n \text{ is even and the other is even or } \infty, \\ 5 & \text{if one of } m, n \text{ is odd and the other is odd or } \infty, \\ 2 & \text{otherwise.} \end{cases}$$

Then (X, d) is a b-metric space with coefficient $s = \frac{5}{2}$.

Definition 1.4. [8] *Let (X, d_X) and (Y, d_Y) be two b-metric spaces. A function*

$f : X \to Y$ is a b-continuous at a point $x \in X$, if it is b-sequentially continuous at x. i.e., whenever $\{x_n\}$ is b-convergent to x, fx_n is b-convergent to fx.

The following lemmas are useful in proving our main results.

Lemma 1.5. [5] *Suppose (X, d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is a not Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$. For each $k > 0$, corresponding to m_k, we can choose n_k to be the smallest positive integer such that $d(x_{m_k}, x_{n_k}) \geq \epsilon, d(x_{m_k}, x_{n_k-1}) < \epsilon$ and*

(i) $\lim\limits_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon$ (ii) $\lim\limits_{k\to\infty} d(x_{n_k-1}, x_{m_k}) = \epsilon$

(iii) $\lim\limits_{k\to\infty} d(x_{m_k+1}, x_{n_k}) = \epsilon$ *and* (iv) $\lim\limits_{k\to\infty} d(x_{m_k+1}, x_{n_k-1}) = \epsilon$.

Lemma 1.6. [23] *Suppose (X, d) is a b-metric space with coefficient $s \geq 1$ and $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is a not Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$. For each $k > 0$, corresponding to m_k, we can choose n_k to be the smallest positive integer such that $d(x_{m_k}, x_{n_k}) \geq \epsilon, d(x_{m_k}, x_{n_k-1}) < \epsilon$ and*

(i) $\epsilon \leq \liminf\limits_{k\to\infty} d(x_{m_k}, x_{n_k}) \leq \limsup\limits_{k\to\infty} d(x_{m_k}, x_{n_k}) \leq s\epsilon$

(ii) $\frac{\epsilon}{s} \leq \liminf\limits_{k\to\infty} d(x_{m_k+1}, x_{n_k}) \leq \limsup\limits_{k\to\infty} d(x_{m_k+1}, x_{n_k}) \leq s^2\epsilon$

(iii) $\frac{\epsilon}{s} \leq \liminf\limits_{k\to\infty} d(x_{m_k}, x_{n_k+1}) \leq \limsup\limits_{k\to\infty} d(x_{m_k}, x_{n_k+1}) \leq s^2\epsilon$

(iv) $\frac{\epsilon}{s^2} \leq \liminf\limits_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) \leq \limsup\limits_{k\to\infty} d(x_{m_k+1}, x_{n_k+1}) \leq s^3\epsilon$.

Lemma 1.7. [1] *Let (X, d) be a b-metric space with coefficient $s \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x and y respectively, then we have*

$$\frac{1}{s^2} d(x, y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x, y).$$

In particular, if $x = y$, then we have $\lim\limits_{n\to\infty} d(x_n, y_n) = 0$. Moreover for each $z \in X$ we have

$$\frac{1}{s} d(x, z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s d(x, z)$$

.

In 2015, Khojasteh, Shukla and Radenović [8] introduced simulation function and defined $\mathcal{Z}$-contraction with respect to a simulation function.

Definition 1.8. [16] *A simulation function is a mapping $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty, \infty)$ satisfying the following conditions:*

(ζ_1) $\zeta(0,0) = 0$;

(ζ_2) $\zeta(t,s) < s - t$ for all $s, t > 0$;

(ζ_3) if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l \in (0,\infty)$ then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

Remark 1.9. [3] *Let ζ be a simulation function. If $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l \in (0,\infty)$, then $\limsup_{n\to\infty} \zeta(kt_n, s_n) < 0$ for any $k > 1$.*

The following are examples of simulation functions.

Example 1.10. [3] Let $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty, \infty)$ be defined by

(i) $\zeta(t,s) = \lambda s - t$ for all $t, s \in \mathbb{R}^+$, where $\lambda \in [0,1)$;

(ii) $\zeta(t,s) = \frac{s}{1+s} - t$ for all $s, t \in \mathbb{R}^+$;

(iii) $\zeta(t,s) = s - kt$ for all $t, s \in \mathbb{R}^+$, where $k > 1$;

(iv) $\zeta(t,s) = \frac{1}{1+s} - (1+t)$ for all $s, t \in \mathbb{R}^+$;

(v) $\zeta(t,s) = \frac{1}{k+s} - t$ for all $s, t \in \mathbb{R}^+$ where $k > 1$.

Definition 1.11 [16] *Let (X,d) be a metric space and $f : X \to X$ be a selfmap of X. We say that f is a $\mathcal{Z}$-contraction with respect to ζ, if there exists a simulation function ζ such that*

$$\zeta(d(fx,fy), d(x,y)) \geq 0$$

for all $x, y \in X$.

Theorem 1.12. [16] *Let (X,d) be a complete metric space and $f : X \to X$ be a $\mathcal{Z}$-contraction with respect to a certain simulation function ζ, then for every $x_0 \in X$, the Picard sequence $\{f^n x_0\}$ converges in X and $\lim_{n\to\infty} f^n x_0 = u$(say) in X and u is the unique fixed point of f in X.*

Recently, Olgun, Bicer and Alyildiz [21] proved the following result in complete metric spaces.

Theorem 1.13. [21] *Let (X,d) be a complete metric space and $f : X \to X$ be a selfmap on X. If there exists a simulation function ζ such that*

$$\zeta(d(fx,fy), M(x,y)) \geq 0$$

for all $x, y \in X$*, where* $M(x,y) = \max\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\}$*, then for every* $x_0 \in X$*, the Picard sequence* $\{f^n x_0\}$ *converges in* X *and* $\lim_{n\to\infty} f^n x_0 = u$ *(say) in* X *and* u *is the unique fixed point of* f *in* X*.*

In 2018, Babu, Dula and Kumar [3] extended Theorem 1.13 of [21] to a pair of selpmaps in the setting of b-metric spaces as follows.

Theorem 1.14. [3] *Let* (X,d) *be a complete* b*-metric space with coefficient* $s \geq 1$ *and* $f, g : X \to X$ *be a selfmaps on* X*. If there exists a simulation function* ζ *such that*

$$\zeta(s^4 d(fx,gy), M(x,y)) \geq 0$$

for all $x, y \in X$*, where* $M(x,y) = \max\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy)+d(y,fx)}{2s}\}$*, then* f *and* g *have a unique common fixed point in* X*, provided either* f *or* g *is* b*-continuous.*

The following theorem is due to Kumam, Gopal and Budhia [17].

Theorem 1.15. [17] *Let* (X,d) *be a complete metric space and* $f : X \to X$ *be a selfmap on* X*. If there exists a simulation function* ζ *such that*

$$\frac{1}{2}d(x,fx) < d(x,y) \implies \zeta(d(fx,fy), d(x,y)) \geq 0$$

for all $x, y \in X$*, then for every* $x_0 \in X$*, the Picard sequence* $\{x_n\}$*, where* $x_n = fx_{n-1}$ *for all* $n \in \mathbb{N}$ *converges to the unique fixed point of* f*.*

In 2018, Padcharoen, Kumam, Saipara and Chaipunya [22], proved the following theorem in complete metric spaces.

Theorem 1.16. [22] *Let* (X,d) *be a complete metric space and* $f : X \to X$ *be a selfmap on* X*. If there exists a simulation function* ζ *such that*

$$\frac{1}{2}d(x,fx) < d(x,y) \implies \zeta(d(fx,fy), M(x,y)) \geq 0$$

for all $x, y \in X$*, where* $M(x,y) = \max\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\}$*, then for every* $x_0 \in X$*, the Picard sequence* $\{x_n\}$*, where* $x_n = fx_{n-1}$ *for all* $n \in \mathbb{N}$ *converges to the unique fixed point of* f*.*

Recently, the authors of the present paper, extended the results, namely Theorem 1.15 and Theorem 1.16 to b-metric spaces [4]. Motivated by these works, we extend Theorem 1.15 and Theorem 1.16 to a pair of maps in b-metric spaces.

In Section 2, we introduce Suzuki $\mathcal{Z}$-contraction type (I) maps, Suzuki $\mathcal{Z}$-contraction type (II) maps in b-metric spaces for a pair of selfmaps and provide examples. In Section 3, we prove the existence and uniqueness of common fixed points of Suzuki $\mathcal{Z}$-contraction type (I) and type (II) maps. In Section 4, we draw some corollaries to our results and provide examples in support of our results.

2. Suzuki $\mathcal{Z}$-contraction type maps

In this section, we introduce Suzuki $\mathcal{Z}$-contraction type (I) maps and Suzuki $\mathcal{Z}$-contraction type (II) maps for a pair of selfmaps in b-metric spaces.

Definition 2.1. *Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $f,g : X \to X$ be selfmaps on X. We say that (f,g) is a Suzuki $\mathcal{Z}$-contraction type (I) maps, if there exists a simulation function ζ such that*

$$\frac{1}{2s}\min\{d(x,fx),d(y,gy)\} \leq d(x,y) \text{ implies that } \zeta(s^4 d(fx,gy), M_1(x,y)) \geq 0 \tag{2.1}$$

for all $x,y \in X$, where $M_1(x,y) = \max\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy)+d(y,fx)}{2s}\}$.

Example 2.2. Let $X = (0,1)$ and let $d : X \times X \to \mathbb{R}^+$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ (x+y)^2 & \text{if } x \neq y. \end{cases}$$

Then clearly (X,d) is a b-metric space with coefficient $s = 2$.
We define $f,g : X \to X$ by $f(x) = \frac{x(5+x)}{256}$ and $g(x) = \frac{x}{16(1+x)}$.
We define $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty,\infty)$ by $\zeta(t,s) = \frac{1}{4}s - t$.
Without loss of generality, we assume that $x \leq y$.
We have $\frac{1}{2s}\min\{d(x,fx),d(y,gy)\} = \frac{1}{4}\min\{(x+\frac{x(5+x)}{256})^2, (y+\frac{y}{16(1+y)})^2\} \leq (x+y)^2 = d(x,y)$
$M_1(x,y) = \max\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy)+d(y,fx)}{2s}\}$
$= \max\{(x+y)^2, (x+\frac{x(5+x)}{256})^2, (y+\frac{y}{16(1+y)})^2, \frac{(x+\frac{y}{16(1+y)})^2+(y+\frac{x(5+x)}{256})^2}{4}\}$.
Now we consider
$s^4 d(fx,gy) = 16(\frac{x(5+x)}{256} + \frac{y}{16(1+y)})^2 = \frac{1}{16}(\frac{x(5+x)}{16} + \frac{y}{(1+y)})^2 \leq \frac{1}{16}(\frac{y(5+y)}{16} + \frac{y}{(1+y)})^2 \leq \frac{1}{16}(y+\frac{y}{(1+y)})^2 \leq \frac{1}{4}(x+y)^2 = \frac{1}{4}d(x,y) \leq \frac{1}{4}M_1(x,y)$.
Therefore the pair (f,g) is a Suzuki $\mathcal{Z}$-contraction type (I) maps.

Definition 2.3. *Let (X,d) be a b-metric space with coefficient $s \geq 1$ and $f,g : X \to X$ be selfmaps on X. We say that (f,g) is a Suzuki $\mathcal{Z}$-contraction type (II) maps, if there exists a simulation function ζ such that*

$$\frac{1}{2s}\min\{d(x,fx),d(y,gy)\} \leq d(x,y) \textit{ implies that } \zeta(s^4 d(fx,gy), M_2(x,y)) \geq 0 \tag{2.2}$$

for all $x,y \in X$, where $M_2(x,y) = \max\{d(x,y), \frac{d(y,gy)[1+d(x,fx)]}{1+d(x,y)}, \frac{d(y,fx)[1+d(x,fx)]}{s^2(1+d(x,y))}\}$.

Example 2.4. Let $X = \mathbb{R}^+$ and let d be defined as in Example 2.2.
We define $f,g : X \to X$ by

$f(x) = \begin{cases} \frac{x^2}{256} & \text{if } x \in [0,1) \\ \frac{1}{16} & \text{if } x \in [1,\infty) \end{cases}$ and $g(x) = \begin{cases} \frac{x(1+x)}{512} & \text{if } x \in [0,1) \\ \frac{1}{32} & \text{if } x \in [1,\infty). \end{cases}$.

We define $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty,\infty)$ by $\zeta(t,s) = \frac{1}{4}s - t$.

Without loss of generality, we assume that $y \le x$.

Case (i): $x, y \in [0,1)$.

We have $\frac{1}{2s}\min\{d(x,fx), d(y,gy)\} = \frac{1}{4}\min\{(x+\frac{x^2}{256})^2, (y+\frac{y(1+y)}{512})^2\} \le (x+y)^2 = d(x,y)$.

$M_2(x,y) = \max\{d(x,y), \frac{d(y,gy)[1+d(x,fx)]}{1+d(x,y)}, \frac{d(y,fx)[1+d(x,fx)]}{s^2(1+d(x,y))}\}$

$= \max\{(x+y)^2, \frac{(y+\frac{y(1+y)}{512})^2[1+(x+\frac{x^2}{256})^2]}{1+(x+y)^2}, \frac{(y+\frac{x^2}{256})^2[1+(x+\frac{x^2}{256})^2]}{4(1+(x+y)^2)}\}$.

Now we consider

$s^4 d(fx,gy) = 16(\frac{x^2}{256}+\frac{y(1+y)}{512})^2 = \frac{1}{16}(\frac{x^2}{16}+\frac{y(1+y)}{32})^2 \le \frac{1}{16}(\frac{x^2}{16}+x)^2$

$\le \frac{1}{4}(x+y)^2 = \frac{1}{4}d(x,y) \le \frac{1}{4}M_2(x,y)$

Case (ii): $x, y \in [1,\infty)$.

We have $\frac{1}{2s}\min\{d(x,fx), d(y,gy)\} = \frac{1}{4}\min\{(x+\frac{1}{16})^2, (y+\frac{1}{32})^2\} \le (x+y)^2 = d(x,y)$.

$M_2(x,y) = \max\{d(x,y), \frac{d(y,gy)[1+d(x,fx)]}{1+d(x,y)}, \frac{d(y,fx)[1+d(x,fx)]}{s^2(1+d(x,y))}\}$

$= \max\{(x+y)^2, \frac{(y+\frac{1}{32})^2[1+(x+\frac{1}{16})^2]}{1+(x+y)^2}, \frac{(y+\frac{1}{16})^2[1+(x+\frac{1}{16})^2]}{4(1+(x+y)^2)}\}$.

Now we consider

$s^4 d(fx,gy) = 16(\frac{1}{16}+\frac{1}{32})^2 = \frac{1}{4}(\frac{9}{16}) \le \frac{1}{4}(x+y)^2 = \frac{1}{4}d(x,y) \le \frac{1}{4}M_2(x,y)$.

Case (iii): $x \in [1,\infty), y \in [0,1)$.

We have $\frac{1}{2s}\min\{d(x,fx), d(y,gy)\} = \frac{1}{4}\min\{(x+\frac{1}{16})^2, (y+\frac{y(1+y)}{512})^2\} \le (x+y)^2 = d(x,y)$.

$M_2(x,y) = \max\{d(x,y), \frac{d(y,gy)[1+d(x,fx)]}{1+d(x,y)}, \frac{d(y,fx)[1+d(x,fx)]}{s^2(1+d(x,y))}\}$

$= \max\{(x+y)^2, \frac{(y+\frac{y(1+y)}{512})^2[1+(x+\frac{1}{16})^2]}{1+(x+y)^2}, \frac{(y+\frac{1}{16})^2[1+(x+\frac{1}{16})^2]}{4(1+(x+y)^2)}\}$.

Now we consider

$s^4 d(fx,gy) = 16(\frac{1}{16}+\frac{y(1+y)}{512})^2 = \frac{1}{16}(1+\frac{y(1+y)}{32})^2 \le \frac{1}{4}(x+y)^2 = \frac{1}{4}d(x,y) \le \frac{1}{4}M_2(x,y)$.

Therefore from all the above cases we conclude that the pair (f,g) is a Suzuki $\mathcal{Z}$-contraction type (II) maps.

Remark 2.5. *It is clear from the definition of simulation function that $\zeta(t,s) < 0$ for all $t \ge s > 0$. Therefore*

(i) if the pair (f,g) satisfies (2.1), then

$$\frac{1}{2s}\min\{d(x,fx), d(y,gy)\} \le d(x,y) \text{ implies that } s^4 d(fx,gy) < M_1(x,y),$$

for all $x, y \in X$; and

(ii) if the pair (f, g) satisfies (2.2), then

$$\frac{1}{2s}\min\{d(x, fx), d(y, gy)\} \leq d(x, y) \text{ implies that } s^4 d(fx, gy) < M_2(x, y),$$

for all $x, y \in X$.

3. Main Results

Proposition 3.1. *Let (X, d) be a b-metric space with coefficient $s \geq 1$ and $f, g : X \to X$ be two selfmaps. Assume that the pair (f, g) is a Suzuki $\mathcal{Z}$-contraction type (I) maps. Then u is a fixed point of f if and only if u is a fixed point of g. Moreover, in this case u is unique.*

Proof. Let u be a fixed point of f. i.e., $fu = u$.
Suppose that $gu \neq u$.
We have
$\frac{1}{2s}\min\{d(u, fu), d(u, gu)\} = \frac{1}{2s}\min\{d(u, u), d(u, gu)\} = 0 = d(u, u)$
and hence from the inequality (2.1), we get
$\zeta(s^4 d(fu, gu), M_1(u, u)) \geq 0$, where
$M_1(u, u) = \max\{d(u, u), d(u, fu), d(u, gu), \frac{d(u,gu)+d(u,fu)}{2s}\} = d(u, gu)$.
By using (ζ_2), we have
$0 \leq \zeta(s^4 d(u, gu), M_1(u, u)) < M_1(u, u) - s^4 d(u, gu) = d(u, gu) - s^4 d(u, gu)$,
a contradiction.
Hence $gu = u$, so that u is a common fixed point of f and g.
Similarly, it is easy to see that if u is a fixed point of g then u is a fixed point of f also.
Suppose u and v are two common fixed points of f and g with $u \neq v$.
Since $\frac{1}{2s}\min\{d(u, fu), d(v, gv)\} \leq d(u, v)$ so that from the inequality (2.1), we get
$\zeta(s^4 d(fu, gv), M_1(u, v)) \geq 0$, where
$M_1(u, v) = \max\{d(u, v), d(u, fu), d(v, gv), \frac{d(u,gv)+d(v,fu)}{2s}\} = d(u, v)$.
By using (ζ_2), we have
$0 \leq \zeta(s^4 d(u, v), M_1(u, v)) < M_1(u, v) - s^4 d(u, v) = d(u, v) - s^4 d(u, v)$,
a contradiction.
Therefore $u = v$. Hence f and g have a unique common fixed point in X.

Proposition 3.2. *Let (X, d) be a b-metric space with coefficient $s \geq 1$ and $f, g : X \to X$ be two selfmaps. Assume that the pair (f, g) is a Suzuki $\mathcal{Z}$-contraction type (II) maps. Then u is a fixed point of f if and only if u is a fixed point of g. Moreover, in this case u is unique.*

Proof. Follows as on the similar lines of Proposition 3.1 and hence we omit the proof.

Theorem 3.3. *Let (X,d) be a complete b-metric space with coefficient $s \geq 1$ and (f,g) be a Suzuki $\mathcal{Z}$-contraction type (I) maps. If either f (or) g is b-continuous then f and g have a unique common fixed point in X.*

Proof. Let $x_0 \in X$ be arbitrary. Since $f(X) \subseteq X$ and $g(X) \subseteq X$, there exist $x_1, x_2 \in X$ such that $fx_0 = x_1$ and $gx_1 = x_2$. Similarly there exist $x_3, x_4 \in X$ such that $fx_2 = x_3$ and $gx_3 = x_4$.

In general, we construct a sequence $\{x_n\}$ in X by $fx_{2n} = x_{2n+1}, gx_{2n+1} = x_{2n+2}$ for $n = 0, 1, 2, \ldots$.

Suppose $x_{2n} = x_{2n+1}$ for some n, then $x_{2n} = fx_{2n}$ so that x_{2n} is a fixed point of f. Hence by Proposition 3.1, we have x_{2n} is a fixed point of g also so that x_{2n} is a common fixed point of f and g.

Similarly, if $x_{2n+1} = x_{2n+2}$ for some n. Then x_{2n+1} is a common fixed point of f and g.

Hence without loss of generality, we assume that $x_n \neq x_{n+1}$ for all n.

Suppose n is even. Then $n = 2m, m \in \mathbb{N}$. Since $\frac{1}{2s}\min\{d(x_n, fx_n), d(x_{n+1}, gx_{n+1})\} = \frac{1}{2s}\min\{d(x_{2m}, fx_{2m}), d(x_{2m+1}, gx_{2m+1})\} \leq d(x_{2m}, x_{2m+1})$, it follows from (2.1) that

$$\zeta(s^4 d(fx_{2m}, gx_{2m+1}), M_1(x_{2m}, x_{2m+1})) \geq 0 \tag{3.1}$$

where

$$\begin{aligned} M_1(x_{2m}, x_{2m+1}) &= \max\{d(x_{2m}, x_{2m+1}), d(x_{2m}, fx_{2m}), d(x_{2m+1}, gx_{2m+1}), \\ &\qquad \tfrac{1}{2s}[d(x_{2m}, gx_{2m+1}) + d(x_{2m+1}, fx_{2m})]\} \\ &= \max\{d(x_{2m}, x_{2m+1}), d(x_{2m}, x_{2m+1}), d(x_{2m+1}, x_{2m+2}), \tfrac{d(x_{2m}, x_{2m+2})}{2s}\} \\ &= \max\{d(x_{2m}, x_{2m+1}), d(x_{2m+1}, x_{2m+2})\}. \end{aligned}$$

If $d(x_{2m}, x_{2m+1}) < d(x_{2m+1}, x_{2m+2})$ then $M_1(x_{2m}, x_{2m+1}) = d(x_{2m+1}, x_{2m+2})$.

Therefore, from (3.1), we have

$$\begin{aligned} 0 &\leq \zeta(s^4 d(x_{2m+1}, x_{2m+2}), M_1(x_{2m}, x_{2m+1})) \\ &= \zeta(s^4 d(x_{2m+1}, x_{2m+2}), d(x_{2m+1}, x_{2m+2})) \\ &< d(x_{2m+1}, x_{2m+2}) - s^4 d(x_{2m+1}, x_{2m+2}), \end{aligned}$$

a contradiction.

Therefore $d(x_n, x_{n+1}) \geq d(x_{n+1}, x_{n+2})$ when n is even. (3.2)

Now, if n is odd, $n = 2m+1$, (say), $m \in \mathbb{N}$.

Since

$$\begin{aligned} \tfrac{1}{2s}\min\{d(x_{n+1}, fx_{n+1}), d(x_n, gx_n)\} &= \tfrac{1}{2s}\min\{d(x_{2m+2}, fx_{2m+2}), d(x_{2m+1}, gx_{2m+1})\} \\ &\leq d(x_{2m+2}, x_{2m+1}), \end{aligned}$$

from (2.1), we have

$$\zeta(s^4 d(fx_{2m+2}, gx_{2m+1}), M_1(x_{2m+2}, x_{2m+1})) \geq 0, \tag{3.3}$$

where

$$\begin{aligned} M_1(x_{2m+2}, x_{2m+1}) &= \max\{d(x_{2m+2}, x_{2m+1}), d(x_{2m+2}, fx_{2m+2}), d(x_{2m+1}, gx_{2m+1}), \\ &\qquad \tfrac{1}{2s}[d(x_{2m+2}, gx_{2m+1}) + d(x_{2m+1}, fx_{2m+2})]\} \\ &= \max\{d(x_{2m+2}, x_{2m+1}), d(x_{2m+2}, x_{2m+3}), d(x_{2m+1}, x_{2m+2}), \\ &\qquad \tfrac{d(x_{2m+1}, x_{2m+3})}{2s}\} \\ &= \max\{d(x_{2m+2}, x_{2m+1}), d(x_{2m+2}, x_{2m+3})\}. \end{aligned}$$

If $d(x_{2m+2}, x_{2m+1}) < d(x_{2m+3}, x_{2m+2})$ then $M_1(x_{2m+2}, x_{2m+1}) = d(x_{2m+3}, x_{2m+2})$.
Therefore from (3.3), we have

$$\begin{aligned} 0 &\le \zeta(s^4 d(x_{2m+3}, x_{2m+2}), M_1(x_{2m+2}, x_{2m+1})) \\ &= \zeta(s^4 d(x_{2m+3}, x_{2m+2}), d(x_{2m+3}, x_{2m+2})) \\ &< d(x_{2m+3}, x_{2m+2}) - s^4 d(x_{2m+3}, x_{2m+2}), \end{aligned}$$

a contradiction.
Therefore $d(x_n, x_{n+1}) \ge d(x_{n+1}, x_{n+2})$ when n is odd. (3.4)
From (3.2) and (3.4), it follows that $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of nonnegative reals.
Thus there exists $r \ge 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$.
Suppose that $r > 0$.
By using the condition (ζ_3) with $t_n = d(x_{n+1}, x_{n+2})$ and $s_n = d(x_n, x_{n+1})$, we have
$0 \le \limsup_{n\to\infty} \zeta(s^4 d(x_{n+1}, x_{n+2}), M_1(x_n, x_{n+1})) < 0$,
it is a contradiction.
Therefore

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.5}$$

Next, we prove that $\{x_n\}$ is a b-Cauchy sequence.
For this it is sufficient to show that $\{x_{2n}\}$ is a b-Cauchy sequence.
On the contrary, suppose that $\{x_{2n}\}$ is not b-Cauchy. We consider the following two cases.
Case (i): $s = 1$.
In this case, (X, d) is a metric space. Then by Lemma 1.5 there exist an $\epsilon > 0$ and sequence of positive integers $\{2n_k\}$ and $\{2m_k\}$ with $2n_k > 2m_k \ge k$ such that $d(x_{2m_k}, x_{2n_k}) \ge \epsilon$ and
$d(x_{2m_k}, x_{2n_k-2}) < \epsilon$ satisfying (i)-(iv) of Lemma 1.5.
Suppose that there exists a $k_1 \in \mathbb{N}$ with $k \ge k_1$ such that

$$\frac{1}{2}\min\{d(x_{2m_k}, fx_{2m_k}), d(x_{2n_k-1}, gx_{2n_k-1})\} > d(x_{2m_k}, x_{2n_k-1}). \tag{3.6}$$

On letting as $k \to \infty$ in (3.6) and using (3.5), we get that $\epsilon \le 0$,
a contradiction.
Therefore $\frac{1}{2}\min\{d(x_{2m_k}, fx_{2m_k}), d(x_{2n_k-1}, gx_{2n_k-1})\} \le d(x_{2m_k}, x_{2n_k-1})$ and from

(2.1), we have
$\zeta(d(fx_{2m_k}, gx_{2n_k-1}), M_1(x_{2m_k}, x_{2n_k-1})) \geq 0$, where

$$\begin{aligned} M_1(x_{2m_k}, x_{2n_k-1}) &= \max\{d(x_{2m_k}, x_{2n_k-1}), d(x_{2m_k}, fx_{2m_k}), d(x_{2n_k-1}, gx_{2n_k-1}), \\ &\qquad \tfrac{1}{2}[d(x_{2n_k-1}, fx_{2m_k}) + d(x_{2m_k}, gx_{2n_k-1})]\} \\ &= \max\{d(x_{2m_k}, x_{2n_k-1}), d(x_{2m_k}, x_{2m_k+1}), d(x_{2n_k-1}, x_{2n_k}), \\ &\qquad \tfrac{1}{2}[d(x_{2n_k-1}, x_{2m_k+1}) + d(x_{2m_k}, x_{2n_k})]\}. \end{aligned}$$

On taking limits as $k \to \infty$ and using (3.5), we get
$\lim\limits_{n\to\infty} M_1(x_{2m_k}, x_{2n_k-1}) = \max\{\epsilon, 0, 0, \epsilon\} = \epsilon$.
By using (ζ_3) with $t_n = d(x_{2m_k+1}, x_{2n_k})$ and $s_n = M_1(x_{2m_k}, x_{2n_k-1})$, we have
$0 \leq \limsup\limits_{k\to\infty} \zeta(d(x_{2m_k+1}, x_{2n_k}), M_1(x_{2m_k}, x_{2n_k-1})) < 0$,
it is a contradiction.

Case (ii): $s > 1$.

In this case, by Lemma 1.6, there exist an $\epsilon > 0$ and sequence of positive integers $\{2n_k\}$ and $\{2m_k\}$ with $2n_k > 2m_k \geq k$ such that $d(x_{2m_k}, x_{2n_k}) \geq \epsilon$ and $d(x_{2m_k}, x_{2n_k-2}) < \epsilon$ satisfying (i)-(iv) of Lemma 1.6.
Suppose that there exists a $k_1 \in \mathbb{N}$ with $k \geq k_1$ such that

$$\frac{1}{2s}\min\{d(x_{2m_k}, fx_{2m_k}), d(x_{2n_k-1}, gx_{2n_k-1})\} > d(x_{2m_k}, x_{2n_k-1}). \tag{3.7}$$

On letting limit superior as $k \to \infty$ in (3.7) and using (3.5), we get that $\epsilon \leq 0$, a contradiction.
Therefore
$\frac{1}{2s}\min\{d(x_{2m_k}, fx_{2m_k}), d(x_{2n_k-1}, gx_{2n_k-1})\} \leq d(x_{2m_k}, x_{2n_k-1})$ and from (2.1), we have

$$\zeta(s^4 d(fx_{2m_k}, gx_{2n_k-1}), M_1(x_{2m_k}, x_{2n_k-1})) \geq 0, \tag{3.8}$$

where

$$\begin{aligned} M_1(x_{2m_k}, x_{2n_k-1}) &= \max\{d(x_{2m_k}, x_{2n_k-1}), d(x_{2m_k}, fx_{2m_k}), d(x_{2n_k-1}, gx_{2n_k-1}), \\ &\qquad \tfrac{1}{2s}[d(x_{2n_k-1}, fx_{2m_k}) + d(x_{2m_k}, gx_{2n_k-1})]\} \\ &= \max\{d(x_{2m_k}, x_{2n_k-1}), d(x_{2m_k}, x_{2m_k+1}), d(x_{2n_k-1}, x_{2n_k}), \\ &\qquad \tfrac{1}{2s}[d(x_{2n_k-1}, x_{2m_k+1}) + d(x_{2m_k}, x_{2n_k})]\}. \end{aligned}$$

On taking limit superior as $k \to \infty$ and using (3.5), we get
$\lim\limits_{n\to\infty} M_1(x_{2m_k}, x_{2n_k-1}) \leq \max\{s\epsilon, 0, 0, \frac{s\epsilon}{2}\} = s\epsilon$.
From (3.8), we have

$$\begin{aligned} 0 &\leq \limsup_{k\to\infty} \zeta(s^4 d(fx_{2m_k}, gx_{2n_k-1}), M_1(x_{2m_k}, x_{2n_k-1})) \\ &\leq \limsup_{k\to\infty} [M_1(x_{2m_k}, x_{2n_k-1}) - s^4 d(x_{2m_k+1}, x_{2n_k})] \\ &= \limsup_{k\to\infty} M_1(x_{2m_k}, x_{2n_k-1}) - s^4 \liminf_{k\to\infty} d(x_{m_k+1}, x_{2n_k}) \leq s\epsilon - s^4\tfrac{\epsilon}{s}, \end{aligned}$$

a contradiction.
Therefore by Case (i) and Case (ii), we have $\{x_n\}$ is a b-Cauchy sequence in X. Since X is b-complete, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = u$.
Therefore $x = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} fx_{2n}$ and $x = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} gx_{2n+1}$ so that $\lim_{n\to\infty} fx_{2n} = x = \lim_{n\to\infty} gx_{2n+1}$.
We assume that f is b-continuous.
Since $x_{2n} \to x$ as $n \to \infty$, we have $fx_{2n} \to fx$ as $n \to \infty$.
Now,

$$0 \leq d(x, fx) \leq s[d(x, fx_{2n}) + d(fx_{2n}, fx)] \to 0 \text{ as } n \to \infty.$$

Therefore x is a fixed point of f.
Hence by Proposition 3.1, x is a unique common fixed point of f and g.

Similarly, we can prove that x is a unique common fixed point of f and g whenever g is b-continuous.

Eventhough, the proof of the following theorem is similar to that of Theorem 3.3, we give its proof and show the importance of the rational term $\frac{d(y,fx)[1+d(x,fx)]}{s^2(1+d(x,y))}$ in the inequality (2.2) (Example 4.2).

Theorem 3.4. *Let (X, d) be a complete b-metric space with coefficient $s \geq 1$ and (f, g) be a Suzuki $\mathcal{Z}$-contraction type (II) maps. If either f (or) g is b-continuous then f and g have a unique common fixed point in X.*

Proof. Let $x_0 \in X$ be arbitrary. Since $f(X) \subseteq X$ and $g(X) \subseteq X$, as in the proof of Theorem 3.3, there exists a sequence $\{x_n\}$ in X such that $fx_{2n} = x_{2n+1}, gx_{2n+1} = x_{2n+2}$ for $n = 0, 1, 2, \ldots$.
Without loss of generality, we assume that $d(x_n, x_{n+1}) > 0$ for all n.
Suppose n is even, $n = 2m$, (say), $m \in \mathbb{N}$.
Since $\frac{1}{2s}\min\{d(x_n, fx_n), d(x_{n+1}, gx_{n+1})\} = \frac{1}{2s}\min\{d(x_{2m}, fx_{2m}), d(x_{2m+1}, gx_{2m+1})\}$
$\leq d(x_{2m}, x_{2m+1})$, from (2.2), we have

$$\zeta(s^4 d(fx_{2m}, gx_{2m+1}), M_2(x_{2m}, x_{2m+1})) \geq 0, \tag{3.9}$$

where

$$\begin{aligned}M_2(x_{2m}, x_{2m+1}) &= \max\{d(x_{2m}, x_{2m+1}), \tfrac{d(x_{2m+1},gx_{2m+1})[1+d(x_{2m},fx_{2m})]}{1+d(x_{2m},x_{2m+1})},\\ &\qquad \tfrac{d(x_{2m+1},fx_{2m})[1+d(x_{2m},fx_{2m})]}{s^2(1+d(x_{2m},x_{2m+1}))}\}\\ &= \max\{d(x_{2m}, x_{2m+1}), d(x_{2m+1}, x_{2m+2})\}.\end{aligned}$$

If $d(x_{2m}, x_{2m+1}) < d(x_{2m+1}, x_{2m+2})$ then $M_2(x_{2m}, x_{2m+1}) = d(x_{2m+1}, x_{2m+2})$.
Therefore from (3.9), we have

$$\begin{aligned}0 &\leq \zeta(s^4 d(x_{2m+1}, x_{2m+2}), M_2(x_{2m}, x_{2m+1}))\\ &= \zeta(s^4 d(x_{2m+1}, x_{2m+2}), d(x_{2m+1}, x_{2m+2}))\end{aligned}$$

$< d(x_{2m+1}, x_{2m+2}) - s^4 d(x_{2m+1}, x_{2m+2})$,
a contradiction.
Therefore $d(x_n, x_{n+1}) \geq d(x_{n+1}, x_{n+2})$ when n is even. (3.10)
Now, if n is odd, $n = 2m+1$, (say), $m \in \mathbb{N}$.
Since
$$\frac{1}{2s}\min\{d(x_{n+1}, fx_{n+1}), d(x_n, gx_n)\} = \frac{1}{2s}\min\{d(x_{2m+2}, fx_{2m+2}), d(x_{2m+1}, gx_{2m+1})\} \leq d(x_{2m+2}, x_{2m+1}) = d(x_{n+1}, x_n).$$
From (2.2), we have
$$\zeta(s^4 d(fx_{2m+2}, gx_{2m+1}), M_2(x_{2m+2}, x_{2m+1})) \geq 0, \tag{3.11}$$
where
$$\begin{aligned} M_2(x_{2m+2}, x_{2m+1}) &= \max\{d(x_{2m+2}, x_{2m+1}), \tfrac{d(x_{2m+1}, gx_{2m+1})[1+d(x_{2m+2}, fx_{2m+2})]}{1+d(x_{2m+2}, x_{2m+1})}, \\ &\quad \tfrac{d(x_{2m+1}, fx_{2m+2})[1+d(x_{2m+2}, fx_{2m+2})]}{s^2(1+d(x_{2m+2}, x_{2m+1}))}\} \\ &= \max\{d(x_{2m+2}, x_{2m+1}), \tfrac{d(x_{2m+1}, x_{2m+2})[1+d(x_{2m+2}, x_{2m+3})]}{1+d(x_{2m+2}, x_{2m+1})}, \\ &\quad \tfrac{d(x_{2m+1}, x_{2m+3})[1+d(x_{2m+2}, x_{2m+2})]}{s^2(1+d(x_{2m+2}, x_{2m+3}))}\}. \end{aligned}$$
If $d(x_{2m+2}, x_{2m+1}) < d(x_{2m+3}, x_{2m+2})$ then $M_2(x_{2m+2}, x_{2m+1}) = d(x_{2m+3}, x_{2m+2})$.
Therefore from (3.11), we have
$0 \leq \zeta(s^4 d(x_{2m+3}, x_{2m+2}), M_2(x_{2m+2}, x_{2m+1}))$
$= \zeta(s^4 d(x_{2m+3}, x_{2m+2}), d(x_{2m+3}, x_{2m+2}))$
$< d(x_{2m+3}, x_{2m+2}) - s^4 d(x_{2m+3}, x_{2m+2})$,
a contradiction.
Therefore $d(x_n, x_{n+1}) \geq d(x_{n+1}, x_{n+2})$ when n is odd. (3.12)
From (3.10) and (3.12), it follows that $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of nonnegative reals.
Thus there exists $r \geq 0$ such that $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = r$. Suppose that $r > 0$.
By using the condition (ζ_3) with $t_n = d(x_{n+1}, x_{n+2})$ and $s_n = d(x_n, x_{n+1})$, we have
$0 \leq \limsup\limits_{n\to\infty} \zeta(s^4 d(x_{n+1}, x_{n+2}), M_2(x_n, x_{n+1})) < 0$,
it is a contradiction.
Therefore $r = 0$.
$$\text{i.e.,} \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.13}$$
We now prove that $\{x_n\}$ is a b-Cauchy sequence.
For this it is sufficient to show that $\{x_{2n}\}$ is a b-Cauchy sequence.
On the contrary suppose that $\{x_{2n}\}$ is not b-Cauchy. We now consider the following two cases.
Case (i): $s = 1$.
In this case, (X, d) is a metric space. Then by Lemma 1.5 there exist an $\epsilon > 0$ and

sequence of positive integers $\{2n_k\}$ and $\{2m_k\}$ with $2n_k > 2m_k \geq k$ such that $d(x_{2m_k}, x_{2n_k}) \geq \epsilon$ and $d(x_{2m_k}, x_{2n_k-2}) < \epsilon$ satisfying (i)-(iv) of Lemma 1.5.
Suppose that there exists a $k_1 \in \mathbb{N}$ with $k \geq k_1$ such that

$$\frac{1}{2}\min\{d(x_{2m_k}, fx_{2m_k}), d(x_{2n_k-1}, gx_{2n_k-1})\} > d(x_{2m_k}, x_{2n_k-1}). \tag{3.14}$$

On taking limits as $k \to \infty$ in (3.14) and using (3.13), we get that $\epsilon \leq 0$, a contradiction.
Therefore $\frac{1}{2}\min\{d(x_{2m_k}, fx_{2m_k}), d(x_{2n_k-1}, gx_{2n_k-1})\} \leq d(x_{2m_k}, x_{2n_k-1})$ and from (2.2), we have
$\zeta(d(fx_{2m_k}, gx_{2n_k-1}), M_2(x_{2m_k}, x_{2n_k-1})) \geq 0$, where

$$\begin{aligned} M_2(x_{2m_k}, x_{2n_k-1}) &= \max\{d(x_{2m_k}, x_{2n_k-1}), \tfrac{d(x_{2n_k-1}, gx_{2n_k-1})[1+d(x_{2m_k}, fx_{2m_k})]}{1+d(x_{2m_k}, x_{2n_k-1})}, \\ &\quad \tfrac{d(x_{2n_k-1}, fx_{2m_k})[1+d(x_{2m_k}, fx_{2m_k})]}{(1+d(x_{2m_k}, x_{2n_k-1}))}\} \\ &= \max\{d(x_{2m_k}, x_{2n_k-1}), \tfrac{d(x_{2n_k-1}, x_{2n_k})[1+d(x_{2m_k}, x_{2m_k+1})]}{1+d(x_{2m_k}, x_{2n_k-1})}, \\ &\quad \tfrac{d(x_{2n_k-1}, x_{2m_k+1})[1+d(x_{2m_k}, x_{2m_k+1})]}{(1+d(x_{2m_k}, x_{2n_k-1}))}\}. \end{aligned}$$

On taking limits as $k \to \infty$ and using (3.13), we get that
$\lim_{n\to\infty} M_2(x_{2m_k}, x_{2n_k-1}) = \max\{\epsilon, 0, \frac{\epsilon}{1+\epsilon}\} = \epsilon$.
By using (ζ_3) with $t_n = d(x_{2m_k+1}, x_{2n_k})$ and $s_n = M_2(x_{2m_k}, x_{2n_k-1})$, we have
$0 \leq \limsup_{k\to\infty} \zeta(d(x_{2m_k+1}, x_{2n_k}), M_2(x_{2m_k}, x_{2n_k-1})) < 0$,
a contradiction.

Case (ii): $s > 1$.

In this case, by Lemma 1.6 there exist an $\epsilon > 0$ and sequence of positive integers $\{2n_k\}$ and $\{2m_k\}$ with $2n_k > 2m_k \geq k$ such that $d(x_{2m_k}, x_{2n_k}) \geq \epsilon$ and $d(x_{2m_k}, x_{2n_k-2}) < \epsilon$ satisfying (i)-(iv) of Lemma 1.6.
Suppose that there exists a $k_1 \in \mathbb{N}$ with $k \geq k_1$ such that

$$\frac{1}{2s}\min\{d(x_{2m_k}, fx_{2m_k}), d(x_{2n_k-1}, gx_{2n_k-1})\} > d(x_{2m_k}, x_{2n_k-1}). \tag{3.15}$$

On letting limit superior as $k \to \infty$ in (3.15) and using (3.13), we get that $\epsilon \leq 0$, a contradiction.
Therefore $\frac{1}{2s}\min\{d(x_{2m_k}, fx_{2m_k}), d(x_{2n_k-1}, gx_{2n_k-1})\} \leq d(x_{2m_k}, x_{2n_k-1})$ and from (2.2), we get

$$\zeta(s^4 d(fx_{2m_k}, gx_{2n_k-1}), M_2(x_{2m_k}, x_{2n_k-1})) \geq 0 \tag{3.16}$$

where
$M_2(x_{2m_k}, x_{2n_k-1}) = \max\{d(x_{2m_k}, x_{2n_k-1}), \frac{d(x_{2n_k-1}, gx_{2n_k-1})[1+d(x_{2m_k}, fx_{2m_k})]}{1+d(x_{2m_k}, x_{2n_k-1})},$

$$\frac{d(x_{2n_k-1}, fx_{2m_k})[1+d(x_{2m_k}, fx_{2m_k})]}{s^2(1+d(x_{2m_k}, x_{2n_k-1}))}\}$$
$$= \max\{d(x_{2m_k}, x_{2n_k-1}), \frac{d(x_{2n_k-1}, x_{2n_k})[1+d(x_{2m_k}, x_{2m_k+1})]}{1+d(x_{2m_k}, x_{2n_k-1})},$$
$$\frac{d(x_{2n_k-1}, x_{2m_k+1})[1+d(x_{2m_k}, x_{2m_k+1})]}{s^2(1+d(x_{2m_k}, x_{2n_k-1}))}\}.$$

On taking limit superior as $k \to \infty$ and using (3.13), we get
$\limsup\limits_{n\to\infty} M_2(x_{2m_k}, x_{2n_k-1}) \leq \max\{s^2\epsilon, 0, \frac{s^2\epsilon}{s+\epsilon}\} = s^2\epsilon.$
From the inequality (3.16), we have

$$\begin{aligned}
0 &\leq \limsup_{k\to\infty} \zeta(s^4 d(fx_{2m_k}, gx_{2n_k-1}), M_2(x_{2m_k}, x_{2n_k-1}))\\
&\leq \limsup_{k\to\infty} [M_2(x_{2m_k}, x_{2n_k-1}) - s^4 d(x_{2m_k+1}, x_{2n_k})]\\
&= \limsup_{k\to\infty} M_2(x_{2m_k}, x_{2n_k-1}) - s^4 \liminf_{k\to\infty} d(x_{2m_k+1}, x_{2n_k})\\
&\leq s^2\epsilon - s^4\tfrac{\epsilon}{s},
\end{aligned}$$

a contradiction.
Therefore by Case (i) and Case (ii), we have $\{x_n\}$ is a b-Cauchy sequence in X. Since X is b-complete, there exists $x \in X$ such that $\lim\limits_{n\to\infty} x_n = x$.
Therefore $x = \lim\limits_{n\to\infty} x_{2n+1} = \lim\limits_{n\to\infty} fx_{2n}$ and $x = \lim\limits_{n\to\infty} x_{2n+2} = \lim\limits_{n\to\infty} gx_{2n+1}$ so that $\lim\limits_{n\to\infty} fx_{2n} = x = \lim\limits_{n\to\infty} gx_{2n+1}$.
We assume that f is b-continuous. Since $x_{2n} \to x$ as $n \to \infty$, we have $fx_{2n} \to fx$ as $n \to \infty$.
Hence, $0 \leq d(x, fx) \leq s[d(x, fx_{2n}) + d(fx_{2n}, fx)] \to 0$ as $n \to \infty$.
Therefore x is a fixed point of f.
Hence, by Proposition 3.2, it follows that x is a unique common fixed point of f and g.

Similarly, we can prove that x is a unique common fixed point of f and g whenever g is b-continuous.

4. Examples and corollaries

The following is an example in support of Theorem 3.3.

Example 4.1. Let $X = [0, 1]$. We define $d : X \times X \to \mathbb{R}^+$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y,\\ \frac{11}{15} & \text{if } x, y \in [0, \frac{2}{3}],\\ \frac{23}{25} + \frac{x+y}{26} & \text{if } x, y \in (\frac{2}{3}, 1],\\ \frac{121}{250} & \text{otherwise.} \end{cases}$$

Then clearly (X, d) is a complete b-metric space with coefficient $s = \frac{51}{49}$.
Here we observe that when $x = \frac{9}{10}, z = 1 \in (\frac{2}{3}, 1]$ and $y \in (0, \frac{2}{3}]$, we have $d(x, z) = \frac{23}{25} + \frac{x+z}{26} = \frac{1291}{1300} \nleq \frac{121}{125} = \frac{121}{250} + \frac{121}{250} = d(x, y) + d(y, z)$ so that d is not a metric.

We define $f, g : X \to X$ by

$$f(x) = \begin{cases} x & \text{if } x \in [0, \frac{2}{3}) \\ \frac{4}{3} - x & \text{if } x \in [\frac{2}{3}, 1] \end{cases} \quad \text{and} \quad g(x) = \begin{cases} \frac{x+3}{4} & \text{if } x \in [0, \frac{2}{3}) \\ 1 - \frac{x}{2} & \text{if } x \in [\frac{2}{3}, 1]. \end{cases}$$

Clearly f is b-continuous.
We define $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty, \infty)$ by $\zeta(s, t) = \frac{99}{100}t - s, t \geq 0, s \geq 0$.
Then ζ is a simulation function. Without loss of generality, we assume that $x \geq y$.
Case (i): $x, y \in [0, \frac{2}{3})$.
$\frac{1}{2s}\min\{d(x, fx), d(y, gy)\} = (\frac{49}{102})(\frac{121}{250}) \leq \frac{11}{15} = d(x, y)$.
$d(fx, gy) = \frac{121}{250}, d(x, y) = \frac{11}{15}, d(x, fx) = \frac{11}{15}, d(y, gy) = \frac{121}{250}, d(y, fx) = \frac{11}{15}, d(x, gy) = \frac{121}{250}$.

$$\begin{aligned} M_1(x, y) &= \max\{d(x, y), d(x, fx), d(y, gy), \tfrac{d(x,gy)+d(y,fx)}{2s}\} \\ &= \max\{\tfrac{11}{15}, \tfrac{11}{15}, \tfrac{121}{250}, \tfrac{49[\frac{121}{250}+\frac{11}{15}]}{102}\} = \tfrac{11}{15}. \end{aligned}$$

We now consider
$\zeta(s^4 d(fx, gy), M_1(x, y)) = \frac{99}{100}M_1(x, y) - s^4 d(fx, gy) = \frac{99}{100}(\frac{11}{15}) - (\frac{51}{49})^4(\frac{121}{250}) \geq 0$.
Case (ii): $x, y \in (\frac{2}{3}, 1]$.
$\frac{1}{2s}\min\{d(x, fx), d(y, gy)\} = (\frac{49}{102})(\frac{121}{250}) \leq \frac{23}{25} + \frac{x+y}{26} = d(x, y)$.
$d(fx, gy) = \frac{11}{15}, d(x, y) = \frac{23}{25} + \frac{x+y}{26}, d(x, fx) = \frac{121}{250}, d(y, gy) = \frac{121}{250}, d(y, fx) = \frac{121}{250}$, $d(x, gy) = \frac{121}{250}$.

$$\begin{aligned} M_1(x, y) &= \max\{d(x, y), d(x, fx), d(y, gy), \tfrac{d(x,gy)+d(y,fx)}{2s}\} \\ &= \max\{\tfrac{23}{25} + \tfrac{x+y}{26}, d(x, fx) = \tfrac{121}{250}, d(y, gy) = \tfrac{121}{250}, \tfrac{49[\frac{121}{250}+\frac{121}{250}]}{102}\} = \tfrac{23}{25} + \tfrac{x+y}{26}. \end{aligned}$$

We now consider
$\zeta(s^4 d(fx, gy), M_1(x, y)) = \frac{99}{100}M_1(x, y) - s^4 d(fx, gy) = \frac{99}{100}(\frac{23}{25} + \frac{x+y}{26}) - (\frac{51}{49})^4(\frac{11}{15}) \geq 0$.
Case (iii): $x \in (\frac{2}{3}, 1], y \in [0, \frac{2}{3}]$.
$\frac{1}{2s}\min\{d(x, fx), d(y, gy)\} = (\frac{49}{102})(\frac{121}{250}) \leq \frac{27}{10} = d(x, y)$.
$d(fx, gy) = \frac{121}{250}, d(x, y) = \frac{121}{250}, d(x, fx) = \frac{121}{250}, d(y, gy) = \frac{121}{250}, d(y, fx) = \frac{11}{15}$, $d(x, gy) = \frac{23}{25} + \frac{x+y}{26}$.

$$\begin{aligned} M_1(x, y) &= \max\{d(x, y), d(x, fx), d(y, gy), \tfrac{d(x,gy)+d(y,fx)}{2s}\} \\ &= \max\{\tfrac{121}{250}, \tfrac{121}{250}, \tfrac{121}{250}, \tfrac{49[\frac{23}{25}+\frac{x+y}{26}+\frac{11}{15}]}{102}\} = \tfrac{49[\frac{23}{25}+\frac{x+y}{26}+\frac{11}{15}]}{102}. \end{aligned}$$

Now we consider
$\zeta(s^4 d(fx, gy), M_1(x, y)) = \frac{99}{100}M_1(x, y) - s^4 d(fx, gy) = \frac{99}{100}(\frac{49[\frac{23}{25}+\frac{x+y}{26}+\frac{11}{15}]}{102})$
$-(\frac{51}{49})^4(\frac{121}{250}) \geq 0$.
Case (iv): $x = \frac{2}{3}, y \in [0, \frac{2}{3})$.
$\frac{1}{2s}\min\{d(x, fx), d(y, gy)\} = 0 \leq \frac{121}{250} = d(x, y)$.
$d(fx, gy) = \frac{121}{250}, d(x, y) = \frac{121}{250}, d(x, fx) = 0, d(y, gy) = \frac{121}{250}, d(y, fx) = \frac{121}{250}, d(x, gy) = \frac{23}{25} + \frac{x+y}{26}$.

$$M_1(x, y) = \max\{d(x, y), d(x, fx), d(y, gy), \tfrac{d(x,gy)+d(y,fx)}{2s}\}$$

$= \max\{\frac{121}{250}, 0, \frac{121}{250}, \frac{49[\frac{23}{25}+\frac{x+y}{26}+\frac{121}{250}]}{102}\} = \frac{49[\frac{23}{25}+\frac{x+y}{26}+\frac{121}{250}]}{102}$.

We now consider

$\zeta(s^4 d(fx, gy), M_1(x, y)) = \frac{99}{100} M_1(x, y) - s^4 d(fx, gy) = \frac{99}{100}(\frac{49[\frac{23}{25}+\frac{x+y}{26}+\frac{121}{250}]}{102}) - (\frac{51}{49})^4(\frac{121}{250}) \geq 0$.

From all the above cases we conclude that (f, g) is a pair of Suzuki $\mathcal{Z}$-contraction type (I) maps.

Therefore f and g satisfy all the hypotheses of Theorem 3.3 and $\frac{2}{3}$ is the unique common fixed point of f and g.

The following is an example in support of Theorem 3.4.

Example 4.2. Let $X = \mathbb{R}^+$ and let $d : X \times X \to \mathbb{R}^+$ defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 4 & \text{if } x, y \in [0, 1], \\ 5 + \frac{1}{x+y} & \text{if } x, y \in (1, \infty), \\ \frac{27}{10} & \text{otherwise.} \end{cases}$$

Then clearly (X, d) is a complete b-metric space with coefficient $s = \frac{489}{480}$.

We define $f, g : X \to X$ by

$$f(x) = \begin{cases} x^2 & \text{if } x \in [0, 1) \\ \frac{1}{x^2} & \text{if } x \in [1, \infty) \end{cases} \quad \text{and } g(x) = \begin{cases} 2x^2 + 2 & \text{if } x \in [0, 1) \\ \frac{x^2+1}{2} & \text{if } x \in [1, \infty). \end{cases}$$

Clearly f is b-continuous.

We define $\zeta : \mathbb{R}^+ \times \mathbb{R}^+ \to (-\infty, \infty)$ by $\zeta(s, t) = \frac{99}{100} t - s, t \geq 0, s \geq 0$.

Then ζ is a simulation function. Without loss of generality, we assume that $x \geq y$.

Case (i): $x, y \in [0, 1)$.

$\frac{1}{2s} \min\{d(x, fx), d(y, gy)\} = (\frac{480}{978})(\frac{27}{10}) \leq 4 = d(x, y)$.

$d(fx, gy) = \frac{27}{10}, d(x, fx) = 4, d(y, gy) = \frac{27}{10}, d(x, gy) = \frac{27}{10}, d(y, fx) = 4$.

$M_2(x, y) = \max\{d(x, y), \frac{d(y,gy)[1+d(x,fx)]}{1+d(x,y)}, \frac{d(y,fx)[1+d(x,fx)]}{s^2(1+d(x,y))}\}$

$= \max\{4, \frac{27}{10}, \frac{4}{(\frac{489}{480})^2}\} = 4$.

Now we consider

$\zeta(s^4 d(fx, gy), M_2(x, y)) = \frac{99}{100} M_2(x, y) - s^4 d(fx, gy) = \frac{99}{100}(4) - (\frac{489}{480})^4(\frac{27}{10}) \geq 0$.

Case (ii): $x, y \in (1, \infty)$.

$\frac{1}{2s} \min\{d(x, fx), d(y, gy)\} = (\frac{480}{978})(\frac{27}{10}) \leq 5 + \frac{1}{x+y} = d(x, y)$.

$d(fx, gy) = \frac{27}{10}, d(x, y) = 5 + \frac{1}{x+y}, d(x, fx) = \frac{27}{10}, d(y, gy) = 5 + \frac{1}{x+y}, d(y, fx) = \frac{27}{10}$, $d(x, gy) = 5 + \frac{1}{x+y}$.

$M_2(x, y) = \max\{d(x, y), \frac{d(y,gy)[1+d(x,fx)]}{1+d(x,y)}, \frac{d(y,fx)[1+d(x,fx)]}{s^2(1+d(x,y))}\}$

$= \max\{5 + \frac{1}{x+y}, \frac{(5+\frac{1}{x+y})[1+\frac{27}{10}]}{6+\frac{1}{x+y}}, \frac{\frac{27}{10}[1+\frac{27}{10}]}{(\frac{489}{480})^2(6+\frac{1}{x+y})}\} = 5 + \frac{1}{x+y}$.

We now consider

$\zeta(s^4 d(fx, gy), M_2(x, y)) = \frac{99}{100} M_2(x, y) - s^4 d(fx, gy) = \frac{99}{100}(5 + \frac{1}{x+y}) - (\frac{489}{480})^4(\frac{27}{10}) \geq 0$.

Case (iii): $x \in (1, \infty), y \in [0, 1)$.
$\frac{1}{2s}\min\{d(x, fx), d(y, gy)\} = (\frac{480}{978})(\frac{27}{10}) \leq \frac{27}{10} = d(x, y)$.
$d(fx, gy) = \frac{27}{10}, d(x, y) = \frac{27}{10}, d(x, fx) = \frac{27}{10}, d(y, gy) = \frac{27}{10}, d(y, fx) = 4, d(x, gy) = 5 + \frac{1}{x+y}$.

$$M_2(x, y) = \max\{d(x, y), \frac{d(y,gy)[1+d(x,fx)]}{1+d(x,y)}, \frac{d(y,fx)[1+d(x,fx)]}{s^2(1+d(x,y))}\}$$
$$= \max\{\frac{27}{10}, \frac{\frac{27}{10}[1+\frac{27}{10}]}{1+\frac{27}{10}}, \frac{4[1+\frac{27}{10}]}{(\frac{489}{480})^2(1+\frac{27}{10})}\} = \frac{4}{(\frac{489}{480})^2}.$$

We now consider
$\zeta(s^4 d(fx, gy), M_2(x, y)) = \frac{99}{100}M_2(x, y) - s^4 d(fx, gy) = \frac{99}{100}(\frac{4}{(\frac{489}{480})^2}) - (\frac{489}{480})^4(\frac{27}{10}) \geq 0$.

Case (iv): $x = 1, y \in [0, 1)$.
$\frac{1}{2s}\min\{d(x, fx), d(y, gy)\} = 0 \leq 4 = d(x, y)$.
$d(fx, gy) = \frac{27}{10}, d(x, y) = 4, d(x, fx) = 0, d(y, gy) = \frac{27}{10}, d(y, fx) = 4, d(x, gy) = \frac{27}{10}$.

$$M_2(x, y) = \max\{d(x, y), \frac{d(y,gy)[1+d(x,fx)]}{1+d(x,y)}, \frac{d(y,fx)[1+d(x,fx)]}{s^2(1+d(x,y))}\}$$
$$= \max\{4, \frac{27}{50}, \frac{4}{(\frac{489}{480})^2(5)}\} = 4.$$

Now we consider
$\zeta(s^4 d(fx, gy), M_2(x, y)) = \frac{99}{100}M_2(x, y) - s^4 d(fx, gy) = \frac{99}{100}(4) - (\frac{489}{480})^4(\frac{27}{10}) \geq 0$.

From all the above cases (f, g) is a pair of Suzuki $\mathcal{Z}$-contraction type (II) maps. Therefore f and g satisfy all the hypotheses of Theorem 3.4 and 1 is the unique common fixed point of f and g.

Here we observe from Case (iii) that, if we omit the term $\frac{d(y,fx)[1+d(x,fx)]}{s^2(1+d(x,y))}$ from the inequality (2.2), then the inequality (2.2) fails to hold.

For, we choose $x = 2, y = \frac{1}{2}$. In this case

$$\frac{1}{2s}\min\{d(x, fx), d(y, gy)\} = (\frac{480}{978})\min\{d(2, \frac{1}{4}), d(\frac{1}{2}, \frac{5}{2})\} = (\frac{480}{978})\min\{\frac{27}{10}, \frac{27}{10}\}$$
$$= (\frac{480}{978})(\frac{27}{10}) \leq \frac{27}{10} = d(x, y).$$

Here

$$M_2(x, y) = \max\{d(x, y), \frac{d(y,gy)[1+d(x,fx)]}{1+d(x,y)}\} = \max\{d(2, \tfrac{1}{2}), \frac{d(\frac{1}{2},\frac{5}{2})[1+d(2,\frac{1}{4})]}{1+d(2,\frac{1}{2})}\}$$
$$= \max\{\frac{27}{10}, \frac{\frac{27}{10}[1+\frac{27}{10}]}{1+\frac{27}{10}}\} = \frac{27}{10} \text{ and}$$

$d(fx, fy) = d(\frac{1}{4}, 4) = \frac{27}{10}$.
Now
$\zeta(s^4 d(fx, gy), M_2(x, y)) = kM_2(x, y) - s^4 d(fx, gy) = k(\frac{27}{10}) - (\frac{489}{480})^4(\frac{27}{10}) \ngeq 0$ for any $k \in [0, 1)$.

Hence the term $\frac{d(y,fx)[1+d(x,fx)]}{s^2(1+d(x,y))}$ plays an important role in the inequality (2.2).

Corollary 4.3. *Let (X, d) be a b-metric space with coefficient $s \geq 1$. Let $f, g : X \to X$ be two selfmaps on X. Assume that there exist two continuous functions $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\varphi(t) < t \leq \psi(t)$ for all $t > 0$ and $\varphi(t) = \psi(t) = 0$ if and only* *if* $t = 0$ *such that*

$$\frac{1}{2s}\min\{d(x,fx), d(y,gy)\} \leq d(x,y) \text{ implies that } \psi(s^4 d(fx,gy)) \leq \varphi(M_1(x,y)) \tag{4.1}$$

for all $x, y \in X$, *where* $M_1(x,y) = \max\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy)+d(y,fx)}{2s}\}$. *If either* f *(or)* g *is b-continuous then* f *and* g *have a unique common fixed point in* X.

Proof. We choose $\zeta(t,s) = \varphi(s) - \psi(t)$ for all $t, s \in \mathbb{R}^+$. Then ζ is a simulation function. Also, the inequality (4.1) implies the inequality (2.1) holds with this simulation function ζ. Hence by Theorem 3.3, the conclusion of this corollary follows.

Similar to Corollary 4.3, we have the following corollary to Theorem 3.4.

Corollary 4.4. *Let* (X, d) *be a b-metric space with coefficient* $s \geq 1$. *Let* $f, g : X \to X$ *be two selfmaps on* X. *Assume that there exist two continuous functions* $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\varphi(t) < t \leq \psi(t)$ *for all* $t > 0$ *and* $\varphi(t) = \psi(t) = 0$ *if and only if* $t = 0$ *such that*

$$\frac{1}{2s}\min\{d(x,fx), d(y,gy)\} \leq d(x,y) \text{ implies that } \psi(s^4 d(fx,gy)) \leq \varphi(M_2(x,y))$$

for all $x, y \in X$, *where* $M_2(x,y) = \max\{d(x,y), \frac{d(y,gy)[1+d(x,fx)]}{1+d(x,y)}, \frac{d(y,fx)[1+d(x,gy)]}{s^2(1+d(x,y))}\}$. *Then* f *and* g *have a unique common fixed point in* X, *provided* f *(or)* g *is b-continuous.*

By choosing $g = f$ in Theorem 3.3 and Theorem 3.4, we have the following corollaries.

Corollary 4.5. [4] *Let* (X, d) *be a complete b-metric space with coefficient* $s \geq 1$ *and* $f : X \to X$ *be a Suzuki* $\mathcal{Z}$*-contraction type* (I) *map. Then* f *has a unique fixed point in* X.

Corollary 4.6. [4] *Let* (X, d) *be a complete b-metric space with coefficient* $s \geq 1$ *and* $f : X \to X$ *be a Suzuki* $\mathcal{Z}$*-contraction type* (II) *map. Then* f *has a unique fixed point in* X.

5. Conclusion

In this paper, we introduced Suzuki $\mathcal{Z}$-contraction type (I) maps, Suzuki $\mathcal{Z}$-contraction type (II) maps, for a pair of selfmaps in b-metric spaces and proved the existence and uniqueness of common fixed points. Our results extend/generalize the known results that are available in the literature. We provided examples in support of our results and some corollaries to our results are presented.

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